[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 8897 (14) Your Roll No. 2019

Unique Paper Code : 235104

~~Paper Code : MAHT 103~~

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Algebra

Semester : I

Duration: 3 Hours Maximum Marks: 75

Instruction for Candidates

1) All six questions are compulsory.
2) Do any two parts from each question.
3) Marks for each part of a question are written against the question in the margin.

1. a) Find the polar representation of complex number

$$z = \cos a - i \sin a,\ a \in [0,2\pi)$$

b) Compute the following

$$z^n + \frac{1}{z^n},\ \text{if } z + \frac{1}{z} = \sqrt{3}.$$

c) Find the quadratic equation whose roots are the cubes of the roots of the equation $x^2 - px + q = 0$.

2. a) For $a, b \in \mathbb{R} \setminus \{0\}$, define $a \sim b$ if and only if $\frac{a}{b} \in \mathbb{Q}$

i. Prove that $\sim$ defines an equivalence relation.

ii. What is an equivalence class of 1? Show that $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.

b) Given three consecutive integers $a, a+1, a+2$, prove that one of them is divisible by 3.

c) ~~a)~~ Define $f: \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = 3x^3 + x.$$

Determine whether or not f is one to one and/or onto.

3. a) Use mathematical induction to establish that for all $n \geq 1$, $8^n - 3^n$ is divisible by 3.

b) Show that the set of rational numbers is countable.

c) Find all integers $x, 0 \le x < 6$, satisfying the following congruence $4x \equiv 2 \pmod 6$.

4. a) Find the general solution of the system

$$\begin{aligned} 2x_1 - x_2 + x_3 + 2x_4 &= 0 \\ -2x_1 + 4x_2 - x_3 + 2x_4 &= -5 \\ x_1 - 6x_2 + 3x_3 + x_4 &= 7 \\ 4x_1 - 6x_2 + x_3 - 4x_4 &= 9 \end{aligned}$$

by reducing the coefficient matrix to echelon form.

b) Determine whether $\boldsymbol{b}$ belongs to the linear span of $\boldsymbol{a_1}, \boldsymbol{a_2}$ and $\boldsymbol{a_3}$, where

$$\boldsymbol{a_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a_2} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}, \boldsymbol{a_3} = \begin{pmatrix} -6 \\ 7 \\ 5 \end{pmatrix}, \text{ and } \boldsymbol{b} = \begin{pmatrix} 11 \\ -5 \\ 9 \end{pmatrix}.$$

c) Balance the following chemical equation

$$NaHCO_3 + H_3C_6H_5O_7 \dashrightarrow Na_3C_6H_5O_7 + H_2O + CO_2.$$

5. a) For what values of h the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ given below

$$\boldsymbol{v_1} = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}, \boldsymbol{v_2} = \begin{pmatrix} -6 \\ 4 \\ -3 \end{pmatrix}, \boldsymbol{v_3} = \begin{pmatrix} 9 \\ h \\ 3 \end{pmatrix},$$

are linearly dependent?

b) Let $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{y_1} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, and $\boldsymbol{y_2} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be suc that $T\boldsymbol{e_1} = \boldsymbol{y_1}$ and $T\boldsymbol{e_2} = \boldsymbol{y_2}$. Find $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

c) (i) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that T is one-to-one i and only if $T(\boldsymbol{x}) = \boldsymbol{0}$ has only the trivial solution.
(ii) Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x_1, x_2) = (x_1 + x_2, x_2)$ is one-to-one

6. a) Find the standard matrix of the horizontal sheer transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ leaves $\boldsymbol{e_1}$ unchanged and maps $\boldsymbol{e_2}$ into $\boldsymbol{e_2} + 2\boldsymbol{e_1}$.

b) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transform and A be standard matrix representatio T. Show that T is invertible linear transformation if and only if A is an inver matrix.

c) Determine the rank of the matrix

$$\begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{pmatrix}.$$